it is seen that in this set are an infinity of numbers greater than b and that they approach b as a limit as p increases indefinitely.

In the fourth quadrant the situation is precisely similar to that in the second.

In the third quadrant $\theta = \pi$, $\phi = \pi$, and (5b) becomes

$$(2p + 1)y = (2q + 1)x + 2n. \tag{9}$$

Hence the points of C_3 on $x^y = y^x$ in the vicinity of any point of C_3 form a dense point aggregate. For in this case we need consider only the lines when $q = p$,

$$y = x + 2n/(2p + 1),$$

which are parallel and have intercepts which are dense on the y-axis. Since C_3 is nowhere parallel to these lines the conclusion is obvious.

The points $(-2, -4)$ and $(-4, -2)$, in particular, lie on $x^y = y^x$.

Conclusion.—Fig. 4 is the graph of $\dfrac{\text{Log}\,|\,x\,|}{x} = \dfrac{\text{Log}\,|\,y\,|}{y}$. It contains all points of the graph of $x^y = y^x$; also it contains other points. Every point on the line $y = x$ and every point on the curve C_1 lies on $x^y = y^x$; the points H and J are not on the locus, but there is an everywhere dense set of points on C_1, C_2, C_3 lying on the locus. I do not know whether there are points on C_3 which are not on $x^y = y^x$, or how many points on C_2 and C_4 are not on $x^y = y^x$.

CONCERNING ROULETTES.

By GOLDIE HORTON, University of Texas.

I. If one curve rolls on another the curve traced by any point in the plane of the rolling curve is called a roulette. The rolling curve is called the moving centrode M and the fixed curve is called the fixed centrode F. To safely apply the methods of infinitesimals we shall suppose that the functions employed in defining the moving and fixed centrodes have continuous first and second derivatives.

(1) As M rolls on F, a line l in the plane of M and a point P on l go into a line l' and a point P' on l', and the point of contact T of M and F moves to T'. The range P on l is congruent to the range P' on l' and hence the pencil PT is projective with the pencil $P'T'$. Since the limiting position of the intersection of PT and $P'T'$ as P' moves back to P is the center of curvature at P of the roulette traced by P, we have

THEOREM 1. *The locus of the centers of curvature of the elements described simultaneously by all the points of a line in the plane of the moving centrode, for an infinitesimal movement, is a conic tangent to the moving centrode and also to the fixed centrode at the instantaneous center of rotation.*

Bresse attributes this theorem to Rivals.[1] It was Mannheim, however, who

[1] See *Journal de l'École Polytechnique*, Cahier 35, p. 112.

first wrote out a proof.[1] He proves it by means of projective properties after using a formula of Savary to prove the equality of certain angles.

(2) In case the moving centrode is a conic and the tracing point P is on the conic, the pencils PT and $P'T'$ are again projective. Hence

THEOREM 2. *If the moving centrode is a conic, the locus of the centers of curvature of the elements described simultaneously by the points of the conic, for an infinitesimal movement, is a conic tangent to the rolling conic and to the fixed centrode at the instantaneous center of rotation.*

This proof of this theorem is given by Mannheim in the paper above referred to.

(3) In case the moving centrode is a circle and the tracing point P is on the circle, the angle between any two rays of the pencil PT equals the angle between the corresponding rays of the pencil $P'T'$, and we have

THEOREM 3. *If the moving centrode is a circle, the locus of the centers of curvature of the elements described simultaneously by the points of the circle, for an infinitesimal movement, is a* CIRCLE *tangent to the rolling circle and to the fixed centrode at the instantaneous center of rotation.*

So far as we know this theorem has not been stated before. It is the purpose of this paper to show its importance in the theory of roulettes.

II. In case both centrodes are circles (we shall note in (6) that they can always be so regarded) there exists a very simple relation between their radii and the maximum radius of curvature of the corresponding epi- or hypocycloid. We now derive this relation and show that from it and Theorem 3 there follows a simple proof of Savary's elegant construction of the center of curvature at any point of a roulette.

(1) THEOREM 4. *Let ρ be the radius of the fixed circle, r that of the rolling circle, and R the radius of curvature of the point P on the rolling circle where the diameter through the rolling point cuts the circle on the opposite side. Then for the epicycloid*

$$\frac{\rho}{\rho+2r}=\frac{R-2r}{2r},$$

and for the hypocycloid

$$\frac{\rho}{\rho-2r}=\frac{R-2r}{2r}.$$

To establish the first relation consider the arc δ traced by P as the moving circle rolls over an arc δ' of the fixed circle. First, the arc δ is the product of $2r$ and the angle of rotation, which is the sum of the angles subtended by δ' at the centers of the two circles, that is, $\delta'/r+\delta'/\rho$; second, the arc δ is the product of R and the angle between successive normals to the roulette, which is $\delta'/(R-2r)$. This double expression gives

$$\frac{\rho}{\rho+2r}=\frac{R-2r}{2r}$$

[1] Mannheim, *Construction of Centers of Curvature, ibid.*, Cahier 37, p. 187.

For the hypocycloid ρ changes sign, and the relation becomes

$$\frac{\rho}{\rho - 2r} = \frac{R - 2r}{2r}.$$

(2) In view of Theorem 3 it follows, in either of the above cases, that the circle, which is the locus of the centers of curvature of the elements described simultaneously by the points of the rolling circle, for an infinitesimal movement, is of diameter $R - 2r$.

(3) Theorems 3 and 4 give the following construction for the center of curvature of the epicycloid or hypocycloid corresponding to any point.

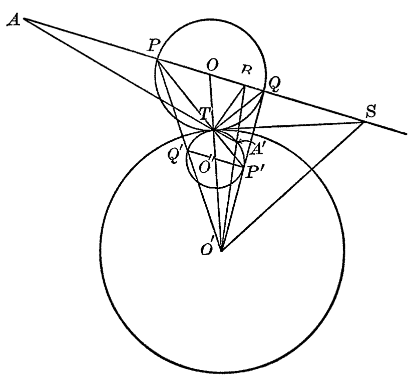

FIG. 1.

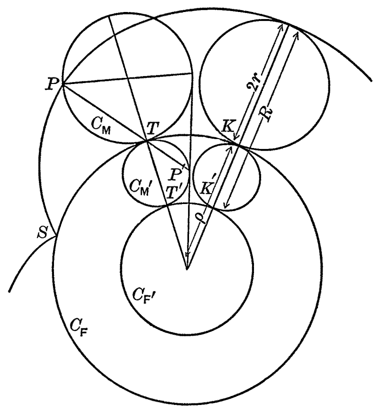

FIG. 2.

Consider the epicycloid. Let P be a point on the rolling circle (see Fig. 1). Draw PT, TQ perpendicular to PT, $O'Q$ and thus determine P'. Determine Q' similarly. Now P' is the center of curvature of the element at P of the epicycloid traced by P, and Q' is the center of curvature of the element at Q of the epicycloid traced simultaneously by Q. That is, P' and Q' are on the circle which is the locus of the centers of curvature of the elements described simultaneously by the points of the moving circle, for an infinitely small movement, and which by (2) above has for diameter $R - 2r$. For, from elementary geometry, O'' is the center of the circle described on $P'Q'$ as diameter. Also

$$\frac{P'Q'}{PQ} = \frac{\rho}{2r + \rho},$$

or,

$$P'Q' = \frac{2\rho r}{2r + \rho},$$

which by Theorem 4 is $R - 2r$; and this proves the construction.

That this construction holds for the hypocycloid may be proved similarly.

(4) We now prove that the center of curvature of the element of a roulette

described by any point A in the plane of the rolling circle is given by the following construction due to Savary. The accompanying figure shows the circles convex to each other. A proof similar to the following shows that the construction holds in the contrary case.

Draw AT, TB (see Fig. 1) perpendicular to AT, BO' and thus determine A', which we prove is the center of curvature of the element of the roulette described, by A for an infinitesimal movement. By Theorem 1 the centers of curvature of the elements of the roulettes traced by the points of the line AO, which we shall call l, lie on a conic touching the fixed and rolling circles at their point of contact T. By (3) above P' and Q' are on this conic; O' is evidently the center of curvature of every point of the roulette traced by O. The conic is therefore determined by P', Q', O' and the point of tangency T. Since the pencils TA and TB are congruent, the ranges A and B on l are projective. Then the pencils (1) and (1′) are projective and hence corresponding rays intersect on a conic. This is the conic already determined for TP and $O'Q$, TO and $O'S$, TQ and $O'P$, and the double rays TS and $O'T$ are corresponding rays of the pencils (1) and (1′).

The construction of the preceding paragraph is a special case of this.

(5) It follows from (4) that for any M and F, that is, for any definition of the movement of the points in the plane of M, the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity, for an infinitesimal movement, is a circle tangent to the fixed and moving centrodes at their point of contact, for in that case the corresponding pencils are congruent.

If M is a line tangent to the fixed centrode the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity is a circle having for diameter the radius of curvature of F at the instantaneous center of rotation.

This property is given by Mannheim in the paper already referred to.

(6) It is to be noticed that the above construction gives the center of curvature of the element described by any point in the plane of the moving centrode in any case in which one can construct the centers of curvature of the fixed and moving centrodes at their point of contact.

(7) As an application of Theorem 4 we shall prove the following well-known theorem.

THEOREM 5. *The evolute of any epicycloid is a similar epicycloid.*

By II (2) the centers of curvature of the elements described simultaneously by the points of the moving circle C_M of radius r, for an infinitesimal movement (see Fig. 2), lie on a circle $C_{M'}$ of radius $\frac{1}{2}(R - 2r)$, which by Theorem 4 may be written $\rho r/(2r + \rho)$. The ratio of the radius of $C_{M'}$ to that of C_M is therefore $\rho/(2r + \rho)$. The locus of the centers of curvature of the high points of C_M as it rolls is clearly a circle $C_{F'}$ concentric with the fixed circle C_F and is of radius $\rho - (R - 2r)$, which may be written $\rho^2/(2r + \rho)$. Then the ratio of the radius of $C_{F'}$ to that of C_F is $\rho/(2r + \rho)$. From elementary geometry we may write

$$\text{arc } TP' = \frac{\rho}{2r+\rho} \text{arc } PT = \text{arc } ST,$$

$$\text{arc } P'T' = \text{arc } TP'T' - \text{arc } TP' = \frac{\rho}{2r+\rho} \text{arc } SK^* - \frac{\rho}{2r+\rho} \text{arc } PT$$

$$= \frac{\rho}{2r+\rho} \text{arc } TK = \text{arc } T'K'.$$

This proves that as C_M rolls on C_F, $C_{M'}$ rolls on $C_{F'}$, that is, as P traces an epicycloid so does P'. Now the ratio of the radii of $C_{M'}$ and $C_{F'}$ is

$$\frac{\rho r}{2r+\rho} \div \frac{\rho^2}{2r+\rho} = \frac{r}{\rho},$$

which is the ratio of the radii of C_M and C_F. Therefore the epicycloid traced by P is similar to that traced by P'.

AN ELEMENTARY THEORY OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By EDWARD V. HUNTINGTON, Harvard University.

In most textbooks on the calculus, the proofs of the formulas for differentiating the logarithmic and exponential functions are either confessedly incomplete, or are made to depend on a preliminary study of the complicated function, $y = \lim_{x=\infty} (1 + 1/x)^x$. This function represents one of the most difficult of the indeterminate forms, the study of which would seem more properly to come late in the course, instead of at the beginning. Moreover, the usual treatment passes over altogether too lightly the questions connected with the existence and meaning of the function a^x for irrational values of x—questions which the student can hardly be supposed to have solved satisfactorily in his previous course in algebra.

The present paper is an attempt to develop the theory of logarithms and exponents, including existence theorems and rules for differentiation, in a new way, which it is hoped will prove not only rigorous but teachable. The discussion is confined to the case of the real variable, and *no knowledge of algebra beyond positive integral exponents is pre-supposed.*

Definition of the Exponential Curve. Let us begin by supposing that in the process of plotting a variety of different curves, some one hit upon the idea of plotting the family of curves representing the following simple algebraic functions:

(A) $\quad (1 + x/2)^2,\ (1 + x/4)^4,\ (1 + x/16)^{16},\ \cdots (1 + x/m)^m,\ \cdots$

(B) $\quad \dfrac{1}{(1 - x/2)^2},\ \dfrac{1}{(1 - x/4)^4},\ \dfrac{1}{(1 - x/16)^{16}},\ \cdots \dfrac{1}{(1 - x/m)^m},\ \cdots,$

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* Since SK is equal to a semicircumference of C_M.